Cooperation percolation in spatial prisoner's dilemma game

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Han-Xin Yang[1], **Zhihai Rong**[2] **and Wen-Xu Wang**[3,4]

[1] Department of Physics, Fuzhou University, Fuzhou 350108, People's Republic of China
[2] Web Sciences Center, School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 610054, People's Republic of China
[3] School of Systems Science, Beijing Normal University, Beijing 100875, People's Republic of China
E-mail: wenxuwang@bnu.edu.cn

Abstract

The paradox of cooperation among selfish individuals still puzzles scientific communities. Although a large amount of evidence has demonstrated that the cooperator clusters in spatial games are effective in protecting the cooperators against the invasion of defectors, we continue to lack the condition for the formation of a giant cooperator cluster that ensures the prevalence of cooperation in a system. Here, we study the dynamical organization of the cooperator clusters in spatial prisoner's dilemma game to offer the condition for the dominance of cooperation, finding that a phase transition characterized by the emergence of a large spanning cooperator cluster occurs when the initial fraction of the cooperators exceeds a certain threshold. Interestingly, the phase transition belongs to different universality classes of percolation determined by the temptation to defect b. Specifically, on square lattices, $1 < b < 4/3$ leads to a phase transition pertaining to the class of regular site percolation, whereas $3/2 < b < 2$ gives rise to a phase transition subject to invasion percolation with trapping. Our findings offer a deeper understanding of cooperative behavior in nature and society.

[4] Author to whom any correspondence should be addressed.

1. Introduction

Cooperation is ubiquitous in biological and social systems [1]. Understanding the emergence and persistence of cooperation among selfish individuals remains an outstanding problem. The development of evolutionary game theory has offered a powerful mathematical framework to address this problem [2]. In order to capture the interaction pattern among greedy individuals, various models have been introduced, among which the prisoner's dilemma game (PDG) has been a prevailing paradigm [3].

In the original PDG, two players simultaneously decide whether to cooperate or defect. They both receive the payoff R upon mutual cooperation and the payoff P upon mutual defection. If one cooperates but the other defects, the defector gets the payoff T, while the cooperator gains the payoff S. The payoff rank for the PDG is $T > R > P > S$. As a result, in a single round of PDG, mutual defection is the best strategy for both players, rendering the social dilemma, in other words, the Nash equilibrium. For simplicity, but without loss of generality, the elements of the payoff matrix for the PDG are often rescaled with $T = b > 1$, $R = 1$ and $P = S = 0$ [4], where b denotes the temptation to defect. Since the pioneering work of Nowak and May [4], evolutionary games have been extensively studied in a structured population, including regular lattices [5–16] and complex networks [17–36]. To facilitate cooperation in the spatial PDG, a number of mechanisms have been proposed, such as voluntary participation [37], punishment [38], similarity [39], inhomogeneous activity [40], social diversity [41, 42], dynamical linking [43], asymmetric interaction and replacement graph [44], migration [45–48], in-group favoritism [49], interdependent links [50, 51] and so on.

It has been known that the formation of cooperator clusters plays an important role in promoting cooperation in spatial games [52]. A cooperator cluster is defined as a connected component (subgraph) fully occupied by cooperators. Within clusters, cooperators can assist each other and the benefits of mutual cooperation outweigh the losses against the defector. It is of paramount importance to explore the dynamical organization of the cooperator clusters so as to understand the emergence of cooperation among selfish individuals. Previous studies focused on the size distribution of the cooperator clusters [53, 54], but paid little attention to the conditions under which a giant cooperator cluster of the order of the system size arises. In this paper, we attempt to address this issue from the perspective of percolation theory [55, 56]. In recent years, the percolation transition, characterized by a large spanning cluster arising at the critical point, has been widely found in various dynamical processes, such as the spreading of disease [57], the formation of public opinion [58] and cascading failures [59]. In particular, percolation phenomena with respect to optimal cooperation level in diluted system were discovered in [60, 61]. However, percolation phenomena pertaining to the emergence of cooperator clusters have not yet been studied in the framework of evolutionary games. Here, we report percolation transition behavior characterized by the emergence of a large spanning cooperator cluster when the initial fraction of the cooperators in the system exceeds a certain threshold. The phase transition can be classified to either regular site percolation or invasion percolation with trapping, depending on the temptation to defect. We substantiate our findings by systematic numerical simulations and analysis of the phase transition.

The paper is organized as follows. In section 2, we formalize the problem by introducing the PDG on square lattices. In section 3, we study the cooperation percolation in terms of

different values of the temptation to defect. Finally, conclusions and discussions are presented in section 4.

2. Model and methods

Individuals are located on a $L \times L$ square lattice with periodic boundary conditions. Each individual x can follow one of two strategies: cooperation or defection, described by

$$z_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1}$$

respectively. At each timestep, each individual plays the PDG with its four nearest neighbors. The accumulated payoff of individual x can be expressed as

$$P_x = \sum_{y \in \Omega_x} z_x^{\mathrm{T}} M z_y, \tag{2}$$

where the sum runs over the nearest neighbor set Ω_x of node x and M is the rescaled payoff matrix with

$$M = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}. \tag{3}$$

According to the setting in [62], initially, the cooperation and the defection strategies are randomly assigned to all individuals in terms of some probabilities: with probability f a node occupied by a cooperator and with probability $1 - f$ a node occupied by a defector. The individuals asynchronously update their strategies in a random sequential order [63–66]. A randomly selected player compares its payoff with its nearest neighbors and changes strategy by following the one (including itself) with the highest payoff. After a period of transient time, the system will enter a steady state.

We focus on the formation of the cooperator clusters in the steady state. In order to study the critical behavior regarding the giant cooperator cluster, we employ the normalized size of the largest cooperator cluster s_1, the susceptibility χ and the Binder's fourth-order cumulant U. These quantities are defined as follows [68]:

$$s_1 = \frac{S_1}{N}, \tag{4}$$

$$\chi = N[\langle s_1^2 \rangle - \langle s_1 \rangle^2], \tag{5}$$

$$U = 1 - \frac{\langle s_1^4 \rangle}{3 \langle s_1^2 \rangle^2}, \tag{6}$$

where S_1 is the size of the largest cooperator cluster, $N = L \times L$ is the system size and $\langle \cdots \rangle$ stands for the configurational averages. According to the standard finite-size scaling approach [68], the phase transition point can be identified by Binder's fourth-order cumulant U and there are the following power-law relationships at the critical value:

$$s_1 \sim N^{-\beta/\nu} \quad \text{and} \quad \chi \sim N^{\gamma/\nu}. \tag{7}$$

We can use the tool to identify the phase transitions pertaining to the formation of the cooperator clusters and explore the class to which the phase transition belongs.
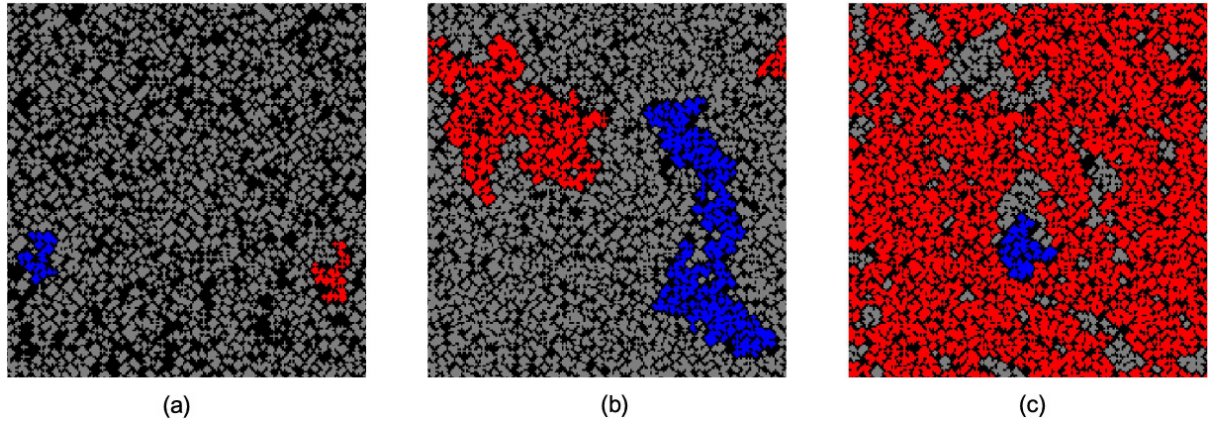
Figure 1. Snapshots of the distributions of the cooperators and the defectors on a 200×200 square lattice. The temptation to defect $1 < b < 4/3$. Red denotes the largest cooperator cluster, blue denotes the second largest cooperator cluster, gray denotes the other cooperator clusters and black denotes the defectors. The initial fraction of cooperators is (a) $f = 0.6$, (b) $f = 0.72$ and (c) $f = 0.76$, respectively.

3. Main results

According to our analysis of the phase transition, we can separate the values of b into four regions: $b \in (1, 4/3)$, $(4/3, 3/2)$, $(3/2, 2)$ and $(2, \infty)$. We find that in the same region, the cooperator clusters with respect to the percolation transition remain unchanged, irrespective of the value of b. In the following, we present the results in the four regions respectively.

3.1. The case of $1 < b < 4/3$

We first explore the characteristics of the cooperator clusters when $1 < b < 4/3$. Figure 1 shows snapshots of the spatial distributions of the cooperators and the defectors on a 200×200 square lattice. We can observe that the size of the largest cooperator cluster (denoted by red) expands as the initial fraction of cooperators f increases. In contrast, the size of the second largest cooperator cluster (denoted by blue) first becomes larger but then shrinks as f continuously increases. The non-monotonic relationship between the size of the second largest cooperator cluster and f implies the existence of a second-order phase transition [67].

Figure 2 shows the normalized size of the largest cooperator cluster s_1 and the fraction F_c of cooperators in the population, as a function of f. We see that both F_c and s_1 increase toward 1 as f increases. Over a wide range of f, F_c is larger than s_1, whereas for very small and large values of f, F_c is approximately equal to s_1. In figure 2, we can also find that there exists a critical value f_c, below which s_1 approaches 0, while above it s_1 continuously increases as f increases. The fact that $F_c > 0.5$ at f_c suggests that it is possible for the cooperators to form a giant cluster comparative to the size of the system, insofar as sufficient numbers of cooperators survive during the evolution.

Figure 3 shows the normalized size of the largest cooperator cluster s_1, the susceptibility χ and the reduced Binder's fourth-order cumulant U as a function of f for different lattice sizes L. Figure 3(a) shows that s_1 is almost the same for different values of L for large f. However, when f is below some value, s_1 decreases as L increases. Figure 3(b) shows that the susceptibility χ
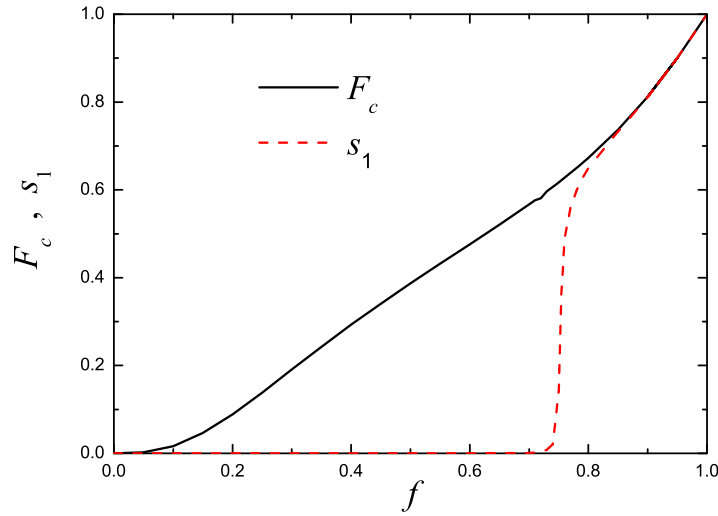
Figure 2. The normalized size of the largest cooperator cluster s_1 and the fraction of the cooperators in the whole population F_c, as a function of the initial fraction of the cooperators f on a 1600×1600 square lattice. The temptation to defect $1 < b < 4/3$. Each curve is an average of 1000 different realizations.
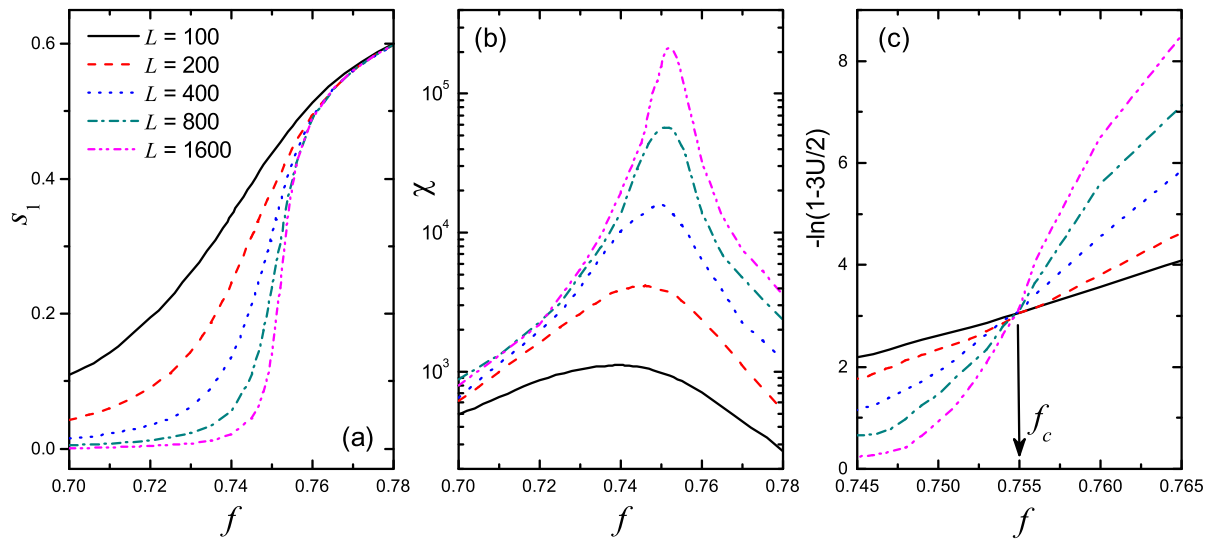

Figure 3. The normalized size of the largest cooperator cluster s_1 (a), the susceptibility χ (b) and the reduced Binder's fourth-order cumulant U (c) as a function of f for different lattice sizes L. In (c), all the curves intersect at $f_c \simeq 0.7551$. The temptation to defect $1 < b < 4/3$. Each curve is an average of 20 000, 10 000, 5000, 2000 and 1000 realizations for $L = 100, 200, 400, 800$ and 1600, respectively.

reaches a maximum at some value of f for different values of L. Moreover, one can see that the value of f that corresponds to the peak of χ increases with L. The critical value f_c can then be identified in figure 3(c), where the curves of the reduced fourth-order cumulant U for different L intersect with each other [69, 70]. The intersection point gives $f_c \simeq 0.7551$ for $1 < b < 4/3$.

Figure 4 shows s_1 and χ as a function of N at the critical value f_c. From figures 4(a) and (b), we obtain the critical exponents $\beta/\nu \simeq 0.051$ and $\gamma/\nu \simeq 0.908$, which are very close to the
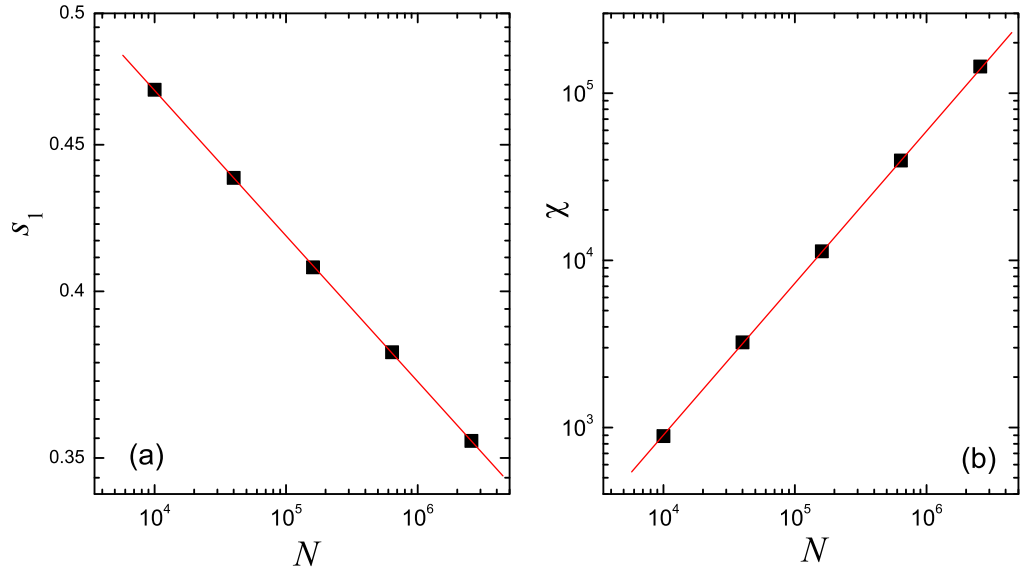
Figure 4. The normalized size of the largest cooperator cluster s_1 (a) and the susceptibility χ (b) as a function of the system size N at the critical value $f_c \simeq 0.7551$. The temptation to defect $1 < b < 4/3$.

critical exponents of the regular site percolation ($\beta/\nu \simeq 0.052$ and $\gamma/\nu \simeq 0.896$) [67]. These results indicate that the cooperation percolation belongs to the same universality class as the regular site percolation when $1 < b < 4/3$.

3.2. The case of $3/2 < b < 2$

We explore the cooperation percolation when $3/2 < b < 2$. Figure 5 shows the normalized size of the largest cooperator cluster s_1, the susceptibility χ and the reduced Binder's fourth-order cumulant U as a function of f for different lattice sizes L. As shown in figure 5(a), s_1 decreases as the system size increases when f is below some value. The susceptibility χ reaches the maximal value at some value of f (see figure 5(b)) and the reduced fourth-order cumulants cross at the critical point $f_c \simeq 0.9553$ (see figure 5(c)).

Figure 6 shows s_1 and χ as a function of the system size N at the critical value f_c. In figures 6(a) and (b), we obtain the critical exponents $\beta/\nu \simeq 0.082$ and $\gamma/\nu \simeq 0.856$, which are close to the critical exponents of the invasion percolation with trapping ($\beta/\nu \simeq 0.084$ and $\gamma/\nu \simeq 0.832$) [71]. These results demonstrate that the cooperation percolation belongs to the same universality class as the invasion percolation with trapping in the region of $3/2 < b < 2$.

3.3. The cases of $4/3 < b < 3/2$ and $b > 2$

Figure 7(a) shows the normalized size of the largest cooperator cluster s_1 and the fraction of co-operators F_c, as a function of the initial concentration of the cooperators f when $4/3 < b < 3/2$. One can observe that F_c is much lower than 1 and $s_1 \approx 0$ even when f is very close to 1. Figure 7(b) shows s_1 as a function of N when initially there is only one defector in the system. We can find that s_1 decreases as N increases. Figure 7 demonstrates that the percolation
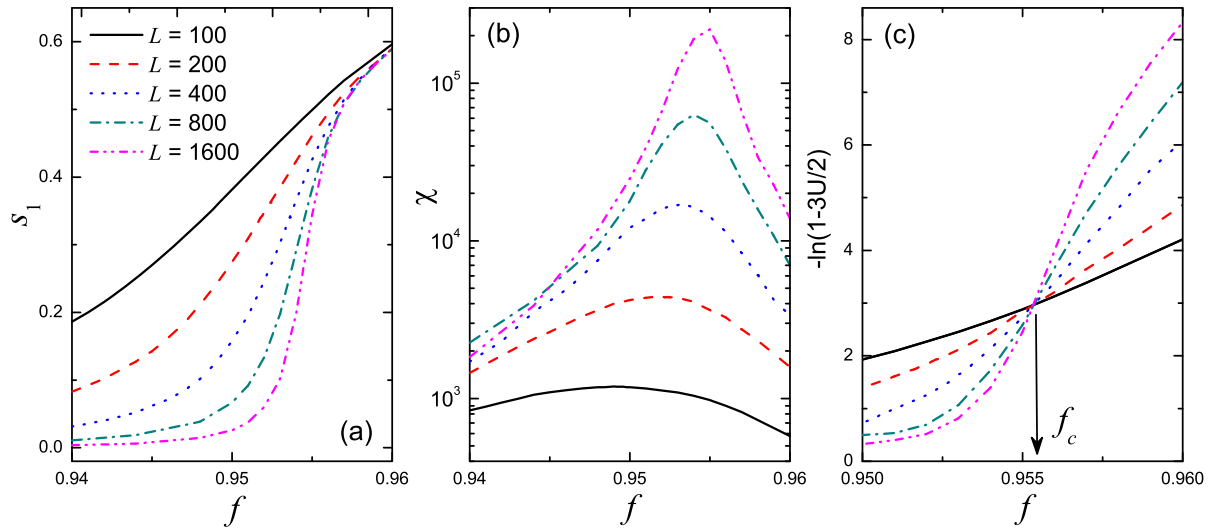
Figure 5. The normalized size of the largest cooperator cluster s_1 (a), the susceptibility χ (b) and the reduced Binder's fourth-order cumulant U (c) as a function of f for different values of the lattice size L. In (c), all the curves intersect at $f_c \simeq 0.9553$. The temptation to defect $3/2 < b < 2$. Each curve is an average of 20 000, 10 000, 5000, 2000 and 1000 realizations for $L = 100, 200, 400, 800$ and 1600, respectively.
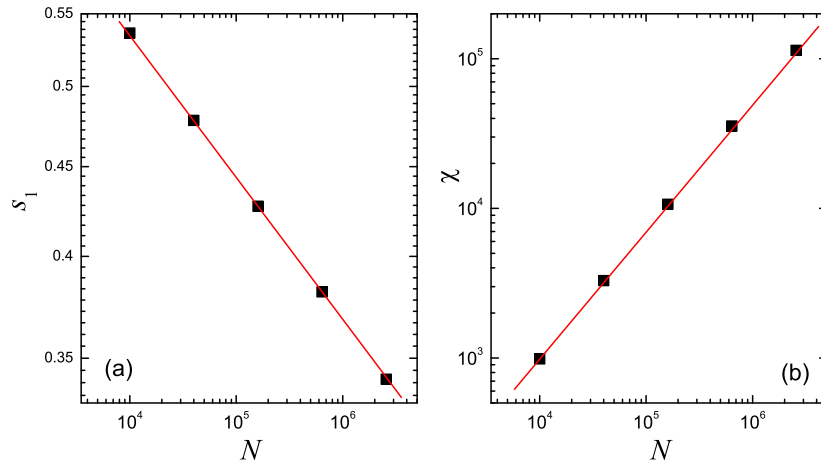

Figure 6. The normalized size of the largest cooperator cluster s_1 (a) and the susceptibility χ (b) as a function of the system size N at the critical value $f_c \simeq 0.9553$. The temptation to defect $3/2 < b < 2$.

threshold for $4/3 < b < 3/2$ is $f_c = 1$. To intuitively understand why the threshold is 1 in the parameter region, we explore the evolution of the spatial distributions of the cooperators and the defectors on a square lattice given a single defector at the center initially ($t = 0$). Figure 8 shows that eventually a giant reticular defector cluster is formed and many small cooperator clusters are separated by the defectors. This result is in agreement with that reported in [72], i.e. the defectors can spread over the system even if initially there is only one defector in the region of $4/3 < b < 3/2$. Hence, only the absence of the defector can lead to a large cooperator cluster that covers the whole lattice, accounting for the phase transition at $f_c = 1$.
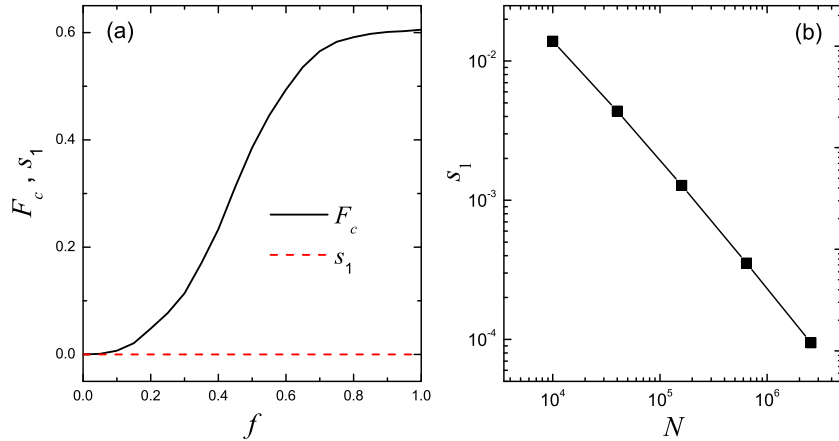
Figure 7. (a) The normalized size of the largest cooperator cluster s_1 and the fraction of the cooperators in the whole population F_c, as a function of the initial fraction of the cooperators f on a 1600×1600 square lattice. (b) The normalized size of the largest cooperator cluster s_1 as a function of the system size N when initially there is only one defector. The temptation to defect $4/3 < b < 3/2$.
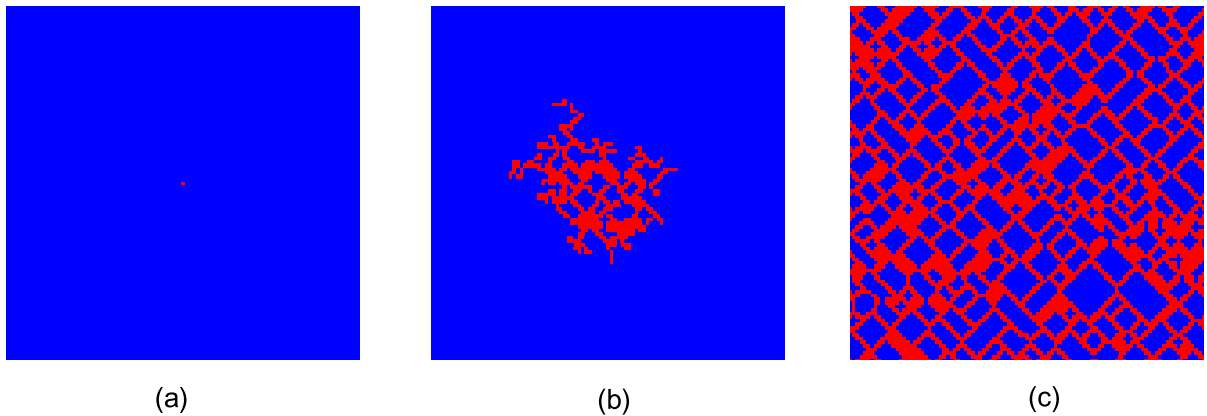

Figure 8. Snapshots of the distributions of the cooperators (blue) and the defectors (red) at different timesteps t on a 99×99 square lattice. The temptation to defect $4/3 < b < 3/2$. (a) $t = 0$, (b) $t = 20$ and (c) $t = 1000$.

In the case of $b > 2$, the cooperators cannot survive when $f < 1$ (the results are not shown here), implying that even one defector can lead to the extinction of the cooperators in the ocean of cooperators when $b > 2$. We can thus infer that the percolation threshold for $b > 2$ is also $f_c = 1$.

4. Conclusions and discussions

In conclusion, we have explored the formation of the cooperator clusters in the PDG, finding that the process of establishing a giant cluster pertains to the percolation behavior. In particular, when the initial fraction of the cooperators in the system exceeds a critical threshold, there arises a giant spanning cooperator cluster, resulting from the merging of many small cooperator clusters. The phase transition behavior is validated by various scaling laws at the critical point, such as

the normalized size of the largest cooperator cluster and the susceptibility scale with the system size. Strikingly, the phase transition belongs to different universality classes, depending on the temptation to defect. The results on the square lattices demonstrate that the phase transition is subject to the class of regular site percolation when $1 < b < 4/3$ or invasion percolation with trapping when $3/2 < b < 2$ whereas in the parameter region $4/3 < b < 3/2$ and $b > 2$, the percolation threshold is 1, indicating that even one defector can prevent the formation of large cooperator clusters. Interestingly, we found that the partition of the parameter region in terms of percolation is exactly the same as the previous findings based on the chaotic spatial patterns in the literature [4, 72]. The agreement offers an underlying connection between the phase transition and the spatial chaos in evolutionary games.

Our findings presented here raise a number of questions, answers to which could further deepen our understanding of the persistence and the dominance of cooperation in terms of a large cooperator cluster. For example, for strategy updating rules rather than the currently used best-take-over rule, such as the Fermi rule [63], does the percolation transition remain? If the answer is positive, will the universality class change? Another significant question pertaining to the network structure is how does the structural property affect the percolation phenomena, e.g. small-world and scale-free topology. Taken together, our results indicate that cooperation in many evolutionary games can be explored from the perspective of the cooperator clusters in combination with the tools for quantifying the phase transition and the percolation, opening new avenues to deepening our understanding of the cooperative behavior widely observed in many aspects in nature and society.

Acknowledgments

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